W29. (Solution by the proposer.) Since $x + y + z = 1$, the given inequality may be written as

$$\sum_{cyclic} \sqrt{\frac{x^3+1}{x^2-x+1}} \leq 3\sqrt{2}.$$

Now, using that $x^3 + 1 = (x+1)(x^2 - x + 1)$, the inequality becomes $\sum_{cyclic} \sqrt{x+1} \leq 3\sqrt{2}$. Finally, since function $f(x) = \sqrt{x+1}$ is concave because $f''(x) = \dfrac{-1}{4(x+1)^{3/2}} < 0$, by Jensen's inequality, we have

$$\frac{\sum_{cyclic} \sqrt{x+1}}{3} \leq \sqrt{\frac{x+y+z}{3} + 1} = \sqrt{2}$$

.

from where the result follows.

Second solution. We have

$$\sum_{cyc}\sqrt{\frac{x^3+1}{x^2+y+z}}=\sum_{cyc}\sqrt{\frac{x^3+1}{x^2-x+1}}=\sum_{cyc}\sqrt{x+1}$$

and since

$$a+b+c\le\sqrt{3\left(a^2+b^2+c^2\right)}, a,b,c\in\mathbb{R}$$

then for $a=\sqrt{x+1}, b=\sqrt{y+1}, c=\sqrt{z+1}$ we obtain

$$\sum_{cyc}\sqrt{x+1}\le\sqrt{3\sum_{cyc}\left(\sqrt{x+1}\right)^2}=\sqrt{3\left(x+y+z+3\right)}=\sqrt{3\cdot 4}=2\sqrt{3}.$$

Arkady Alt

Third solution. We have

$$\sum\sqrt{\frac{x^3+1}{x^2+y+z}}=\sum\sqrt{\frac{x^3+1}{x^2+1-x}}=\sum\sqrt{x+1}\le$$

$$\le\sqrt{3\left(x+y+z+3\right)}=2\sqrt{3}<3\sqrt{2}$$

Nicuşor Zlota

and as addition to published solutions:

https://www.linkedin.com/feed/update/urn:li:activity:6517980571937972224

W29. **Solution by Stéphane Jaubert**.

With $x+y+z=1$ we have $x^2-x+1=x^2+y+z$

$x^3+1=(x+1)(x^2-x+1)=(x+1)(x^2+y+z)$

Therefore $(x^3+1)/(x^2+y+z)=x+1$ and then,

just apply Cauchy-Schwarz with LHS, we obtain:

$$\sum\frac{x^3+1}{x^2+y+z}=\sum\sqrt{x+1}\le\sqrt{(1+1+1)}\cdot\sqrt{(x+1+y+1+z+1)}=2\sqrt{3}.$$